3. All-round

Problem 3.1. Let A be an $N \times N$ positive definite symmetric matrix with $N \geq 2$. Assume that there exists $\varepsilon \in (0,1)$ with

$$tr\, A \leq N + \varepsilon, \quad \det A \geq 1 - \varepsilon.$$

Then there exists a constant C_N depending only on N such that

$$\|A - I\| \leq C_N \sqrt{\varepsilon},$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm, and I is the $N \times N$ identity matrix.

Problem 3.2. Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers. Let $\{e_n\}$ be the standard orthonormal basis of $\ell^2(\mathbb{N})$. Define $T_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T_n(e_m) = e_{nm}. \tag{10}$$

Prove that

(1) The elements in $\{T_n, T_n^*\}$ can commute with the elements in $\{T_m, T_m^*\}$ if and only if $(n, m) = 1$.
(2) Let T be a bounded linear operator which can commute with all T_n, T_n^*, then $T = c \cdot I$ for some constant c.